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THE EIGHTH MEMOIR ON THE THEORY OF SCREWS, SHOWING HOW PLANE GEOMETRY ILLUSTRATES GENERAL PROBLEMS IN THE DYNAMICS OF A RIGID BODY WITH THREE DEGREES OF FREEDOM.

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(PLATE VI.)

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VIII.

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[Read November 12, 1888.]

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I. Introductory.

§ 1. **Preliminary.**—On the 11th April, 1881, I read a brief Paper to the Royal Irish Academy which was published in their *Proceedings*, ser. II., vol. iii., Science, p. 428, and concluded with the words: "The further development of the subject, on which this Paper is a preliminary note, must form the basis of a future and more extensive memoir." Herein I fulfil this intention.

§ 2. **The Problem Stated.**—The rigid body contemplated in the present theory is unspecified as to form, as to size, and as to material, and it may be homogeneous or heterogeneous. The mobility of the body is expressed by saying that it has freedom of the third order. This is equivalent to the

statement that the six independent co-ordinates, which in general connote a position of a body, here reduce to three. It is characteristic of the present theory that the mechanical conditions producing the constraints may be of unspecified character.

We are to study the equilibrium of this body as well as the effect of impulsive forces. We have, however, nothing further to add to the theory of small oscillations given in the preliminary Paper.

§ 3. **A Fundamental Point in the Theory of Screws.**—Let x, y, z denote the Cartesian co-ordinates of a point in the body referred to axes fixed in space. When the body moves into an adjacent position these co-ordinates become, respectively, $x + \delta x$, $y + \delta y$, $z + \delta z$, and we have, by a well-known consequence of the rigidity of the body,

$$\delta x = a + gz - hy,$$

$$\delta y = b + hx - fz,$$

$$\delta z = c + fy - gx,$$

where a, b, c, f, g, h may be regarded as expressing the six generalized co-ordinates of the new position of the body.

As the body has only three degrees of freedom, its position must be capable of specification by three independent co-ordinates, which we shall call θ_1, θ_2, θ_3. The six quantities, a, b, c, f, g, h, must each be a function of θ_1, θ_2, θ_3, so that when the latter are given the former are determined. As all the movements are infinitely small, it is evident that these equations must be linear, and of the type

$$a = A_1\theta_1 + A_2\theta_2 + A_3\theta_3,$$

in which A_1, A_2, A_3 are constants depending on the character of the constraints. We would similarly have

$$b = B_1\theta_1 + B_2\theta_2 + B_3\theta_3,$$

and so on for all the others.

It is a well-known theorem that the new position of the body defined by θ_1, θ_2, θ_3 may be obtained by a twist about a screw of which the axis is defined by the equations

$$\frac{a+gz-hy}{f}=\frac{b+hx-fz}{g}=\frac{c+fy-gx}{p}.$$

The angle through which the body has been rotated is

$$(f^2+g^2+h^2)^{\frac{1}{2}},$$

and the distance of translation is

$$\frac{af+bg+ch}{(f^2+g^2+h^2)^{\frac{1}{2}}},$$

while the pitch of the screw is

$$\frac{af+bg+ch}{f^2+g^2+h^2}.$$

Every distinct set of three quantities, θ_1, θ_2, θ_3, will correlate to a definite position of the rigid body, and to a group of such sets there will be a corresponding group of positions. Let ρ denote a variable parameter, and let us consider the variations of the set,

$$\rho\theta_1, \quad \rho\theta_2, \quad \rho\theta_3,$$

according as ρ varies. To each value of ρ a corresponding position of the rigid body is appropriate, and we thus have the growth of ρ associated with a definite progress of the body through a series of positions. We can give geometrical precision to a description of this movement. The equations to the axis of the screw, as well as the expression of its pitch, only involve the *ratios* of a, b, c, f, g, h. We have also seen that these quantities are each linear and homogeneous functions of θ_1, θ_2, θ_3. If, therefore, we substitute for θ_1, θ_2, θ_3 the more general values

$$\rho\theta_1, \quad \rho\theta_2, \quad \rho\theta_3,$$

the screw would remain unaltered, both in position and in pitch, though the angle of rotation and the distance of translation will each contain ρ as a factor.

Thus we demonstrate that the several positions denoted by the set $\rho\theta_1$, $\rho\theta_2$, $\rho\theta_3$ are all occupied in succession as we twist the body continuously around one particular screw.

§ 4. **A Geometrical Investigation.**—Let A_1 be a standard position of the body, and let P be any marked point of the body initially at P_1: as the body is displaced P will pursue a certain trajectory which, as the motion is small, may be identified with its tangent on which P_n is a point adjacent to P_1. In travelling from P_1 to P_n, P passes through the several positions, P_2, P_{n-1}. In a similar manner every other point, Q, of the rigid body will pass through a series of positions, Q_2, &c., to Q_n. We thus have the points of the body initially at P_1, Q_1, R_1, respectively, and each moves along a straight line through the successive systems of positions P_2, Q_2, R_2, &c., on to the final position P_n, Q_n, R_n. We may thus think of the consecutive positions occupied by the body A_2, A_3, &c., as defined by the groups of points P_1, Q_1, R_1 and P_2, Q_2, R_2, &c. What we have now to show is, that if the body be twisted by a screw motion direct from A_1 to A_n, it will necessarily pass through the series of positions A_2, A_3, &c. It must be remembered that this is hardly to be regarded as an obvious consequence. From the initial position A_1 to the final position A_n, the number of routes are generally infinitely various, but when these situations are infinitely contiguous, and when the body is rigid, it is impossible to pass from A_1 to A_n by any infinitely small movement except by that one which conducts through the several positions, A_2, A_3, &c.

Suppose the body be carried direct by a twist about a screw from the position A_1 to the position A_n. Since this motion is infinitely small, each point of the body will be carried along a straight line, and as P_1 is to be conveyed to P_n, this straight line can be no other than the line P_1P_n. In its progress P_1 will have reached the position P_2, and when it is there the points Q_1, R_1 will each have advanced to certain positions along the lines Q_1Q_n and R_1R_n, respectively. We have to show that the points reached by Q_1 and R_1 can be no other than the points Q_2 and R_2, respectively. To prove this we shall take the case where P_1, Q_1, R_1 are collinear. Suppose that when P_1 has advanced to P_2, Q_1 shall not have reached Q_2, but shall be at

the intermediate point Q_0. (Fig. 1.) Then the line P_1Q_1 will have moved to P_2Q_0, and as R_1 can only be conveyed along R_1R_2, while at the same time

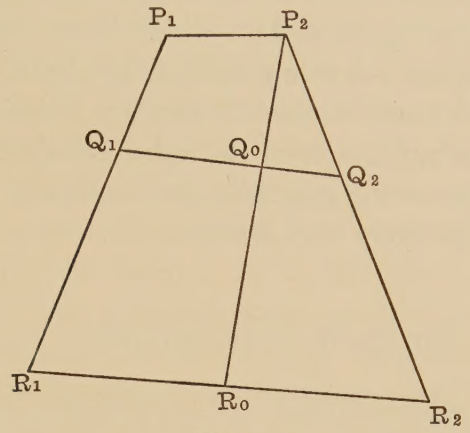


Fig.1.

it must lie along P_2Q_0, it follows that the lines P_2Q_0 and R_1R_2 must intersect at the point R_0, and consequently all the lines in this figure lie in a plane. Further, P_2Q_2 and P_2Q_0 are each equal to P_1Q_1, as the body is rigid, and so also P_2R_0 and P_2R_2 are equal to P_1R_1. Hence it follows that Q_1Q_0 and R_1R_0 are parallel, and consequently all the points on the line $P_1Q_1R_1$ are displaced in parallel directions. It would hence follow that the motion of every point in the body was in a parallel direction, and that consequently the entire movement was simply a translation. But even in this case it would be impossible for the points Q_0 and R_0 to be distinct from Q_2 and R_2, because, when a body is translated so that all its points move in parallel lines, it is impossible, if the body be rigid, for the distances traversed by each point not to be all equal. We have thus demonstrated that if a body is free to move from a position A_1 to an adjacent position A_n by an infinitely small movement, it is also necessarily free to move through the series of positions A_2, A_3, &c., by which it would be conveyed from A_1 to A_n by a twist.

We may also state the matter in a somewhat different manner, as follows:—It would be impossible to devise a system of constraints which would permit a body to be moved somehow from A_1 to A_n, and would at the same time prohibit the body from twisting about the screw which directly

conducts from A_1 to A_n. Of course this would not be true except in the case where the motion is infinitely small. The connexion of this result with the present investigation is now obvious. When A is the standard position of the body, and B any other position into which it can be moved, then the body is free to twist about the screw defined by A and B. Let θ_1 be the angle or amplitude of the twist around this screw. Similarly, let θ_2 and θ_3 be the amplitudes of the twists around two other standard screws, then the three quantities θ_1, θ_2, θ_3 are the generalized co-ordinates for the expression of any position which the body may assume.

II. THE PLANE REPRESENTATION.

§ 5. **The Plane Representation.**—All the possible positions of the body correspond to the triply infinite triads

$$\theta_1, \quad \theta_2, \quad \theta_3,$$

If, for the moment, we regarded these three quantities as the co-ordinates of a point in space, then every point of space will be correlated to a position of the rigid body; but this system of representation leads to nothing elegant or instructive. We shall therefore sort out the triply infinite multitude of positions into a doubly infinite number of groups, each containing a singly infinite number of positions.

If we fix our glance upon the *screws* about which the body is free to twist, the principle of classification will be obvious. Take an arbitrary triad

$$\theta_1, \quad \theta_2, \quad \theta_3,$$

and then form the infinite group of triads

$$\rho\theta_1, \quad \rho\theta_2, \quad \rho\theta_3$$

for every value of ρ from zero up to any finite number: all these triads will correspond to the positions attainable by twisting about a single screw. We may therefore regard

$$\theta_1, \quad \theta_2, \quad \theta_3$$

as the co-ordinates of a screw, it being understood that only the ratios of these quantities are significant.

We are already familiar with a group of three quantities of this nature as the homogeneous, or trilinear co-ordinates of a point in a plane. We thus see that the several screws about which a body with three degrees of freedom can be twisted correspond, severally, with the points of a plane. Think, for a moment, of the boundless extent of a plane, and of the myriads of points which it contains. Yet each one of those points corresponds to a perfectly distinct screw, about which it is possible for a body with three degrees of freedom to be twisted. We shall often speak of the system of screws about which the body can twist as the Three-system, and accordingly we have, as the result of the foregoing discussion, the statement that—

To each screw of a three-system corresponds one point in the plane.

To develop this correspondence is the object of the present Memoir.

§ 6. **The Cylindroid.**—For much that is to follow we shall have to refer to the *Theory of Screws.* In particular, two screws determine a "cylindroid," and the first development of the plane representation will be to show how the screws which constitute the cylindroid are to be represented in the plane. ("Theory," chap. II.). We shall find it convenient to represent the co-ordinates of a screw by the three components into which the unit of twist velocity about that screw may be decomposed along the three screws of reference ("Theory," chap. IV.).

A twist of amplitude θ' has for components on the three screws of reference

$$\theta'\theta_1, \quad \theta'\theta_2, \quad \theta'\theta_3;$$

a twist of amplitude ϕ' on the screw ϕ has the components

$$\phi'\phi_1, \quad \phi'\phi_2, \quad \phi'\phi_3.$$

When these two twists are compounded they will unite into a single twist upon a screw of which the co-ordinates are proportional to

$$\theta'\theta_1 + \phi'\phi_1, \quad \theta'\theta_2 + \phi'\phi_2, \quad \theta'\theta_3 + \phi'\phi_3.$$

If we denote the ratio of ϕ' to θ' by λ, we thus see that the twists about the

screws θ and ϕ unite into a twist about the screw whose co-ordinates are

$$\theta_1 + \lambda\phi_1, \quad \theta_2 + \lambda\phi_2, \quad \theta_3 + \lambda\phi_3;$$

but by the principles of trilinear co-ordinates this point lies on the straight line joining the points θ and ϕ. As the ratio λ varies, the corresponding screw moves over the cylindroid; the corresponding point moves over the straight line. Hence we obtain the following important result:—

The several screws on a cylindroid correspond to the points on a straight line.

In general two cylindroids have no screw in common. If, however, the two cylindroids be each composed of screws taken from the same three-system, then they will have one screw in common. This is demonstrated by the fact that the two straight lines corresponding to these cylindroids necessarily intersect in a point which corresponds to the screw common to the two surfaces.

Three twist velocities about three screws will neutralize and produce rest, provided that the three corresponding points lie in a straight line, and that the amount of each twist velocity is proportional to the sine of the angle between the two non-corresponding screws.

Three wrenches will equilibrate when the three points corresponding to the screws are collinear, and when the intensity of each wrench is proportional to the sine of the angle between the two non-corresponding screws.

III. THE GENERAL SYSTEM OF REPRESENTATIVE CONICS $S + kS' = 0$.

§ 7. **The Screws of the Three-system.**—It will here be necessary to examine the geometrical character of the systems of screws which go to form a three-system. The greater part of this subject has been already given in the *Theory of Screws*, chap. XI. Some further developments are, however, suggested by, and required for, the plane representation discussed in this Paper.

In any three-system there are three principal screws at right angles to each other, and intersecting in a point ("Theory," p. 121). It is natural to choose these as the screws of reference, and also as the axes for Cartesian

Co-ordinates. The pitches of these screws are p_1, p_2, p_3, and we shall, as usual, denote the screw co-ordinates by θ_1, θ_2, θ_3. The displacement denoted by this triad of co-ordinates is obtained by rotating the body through angles θ_1, θ_2, θ_3 around three axes, and then by translating it parallel to these axes through distances $p_1\theta_1$, $p_2\theta_2$, $p_3\theta_3$. As these quantities are all small, we have, for the displacements produced in a point x, y, z,

$$\delta x = p_1\theta_1 + z\theta_2 - y\theta_3,$$

$$\delta y = p_2\theta_2 + x\theta_3 - z\theta_1,$$

$$\delta z = p_3\theta_3 + y\theta_1 - x\theta_2,$$

as already shown (Art. 3), these correspond to a twist about a screw of which the axis has the equations

$$\frac{p_1\theta_1 + z\theta_2 - y\theta_3}{\theta_1} = \frac{p_2\theta_2 + x\theta_3 - z\theta_1}{\theta_2} = \frac{p_3\theta_3 + y\theta_1 - x\theta_2}{\theta_3},$$

while the pitch p is thus given :

$$p = \frac{p_1\theta_1^2 + p_2\theta_2^2 + p_3\theta_3^2}{\theta_1^2 + \theta_2^2 + \theta_3^2}.$$

We have now to investigate the locus of the screws of given pitch, *and as p is presumed to be a determinate quantity*, we have

$$(p_1 - p)\,\theta_1 + z\theta_2 - y\theta_3 = 0,$$

$$+ z\theta_1 + (p_2 - p)\,\theta_2 + x\theta_3 = 0,$$

$$+ y\theta_1 - x\theta_2 + (p_3 - p)\,\theta_3 = 0,$$

whence, by eliminating θ_1, θ_2, θ_3 we obtain, as the locus of the screws of pitch p, the quadric surface

$$(p_1 - p)\,x^2 + (p_2 - p)\,y^2 + (p_3 - p)\,z_2 + (p_1 - p)\,(p_2 - p)\,(p_3 - p) = 0.$$

This family of quadrics, according as p varies, will exhibit all the screws of the three-system which possess a *definite pitch*. To complete the inventory

of the screws it is, however, necessary to add those of *indefinite pitch*, i.e. those whose co-ordinates satisfy the equations

$$p_1\theta_1^2 + p_2\theta_2^2 + p_3\theta_3^2 = 0,$$
$$\theta_1^2 + \theta_2^2 + \theta_3^2 = 0.$$

There are four triads of co-ordinates which satisfy these conditions, and, remembering that only the *ratios* are concerned, the values of θ_1, θ_2, θ_3 may be written thus:

$$\begin{aligned}
&+(p_2-p_3)^{\frac{1}{2}}, \quad +(p_3-p_1)^{\frac{1}{2}}, \quad +(p_1-p_2)^{\frac{1}{2}},\\
&-(p_2-p_3)^{\frac{1}{2}}, \quad +(p_3-p_1)^{\frac{1}{2}}, \quad +(p_1-p_2)^{\frac{1}{2}},\\
&+(p_2-p_3)^{\frac{1}{2}}, \quad -(p_3-p_1)^{\frac{1}{2}}, \quad +(p_1-p_2)^{\frac{1}{2}},\\
&+(p_2-p_3)^{\frac{1}{2}}, \quad +(p_3-p_1)^{\frac{1}{2}}, \quad -(p_1-p_2)^{\frac{1}{2}}.
\end{aligned}$$

The equations of the axis written without p are,

$$x(\theta_2^2+\theta_3^2) - y\theta_1\theta_2 - z\theta_1\theta_3 + (p_2-p_3)\,\theta_2\theta_3 = 0,$$
$$y(\theta_3^2+\theta_1^2) - z\theta_2\theta_3 - x\theta_2\theta_1 + (p_3-p_1)\,\theta_3\theta_1 = 0,$$
$$z(\theta_1^2+\theta_2^2) - x\theta_3\theta_1 - y\theta_3\theta_2 + (p_1-p_2)\,\theta_1\theta_2 = 0,$$

of which two are independent.

If we substitute the values of θ_1, θ_2, θ_3 for the first indeterminate screw, the three equations just written reduce to

$$x(p_2-p_3)^{\frac{1}{2}} + y(p_3-p_1)^{\frac{1}{2}} + z(p_1-p_2)^{\frac{1}{2}} - (p_2-p_3)^{\frac{1}{2}}(p_3-p_1)^{\frac{1}{2}}(p_1-p_2)^{\frac{1}{2}} = 0.$$

The equations of the four planes are briefly expressed, as follows, by making

$$\alpha = (p-p_3)^{\frac{1}{2}}; \quad \beta = (p_3-p_1)^{\frac{1}{2}}; \quad \gamma = (p_1-p_2)^{\frac{1}{2}},$$

$$\begin{aligned}
&+\alpha x + \beta y + \gamma z - \alpha\beta\gamma = 0,\\
&-\alpha x + \beta y + \gamma z - \alpha\beta\gamma = 0,\\
&+\alpha x - \beta y + \gamma z - \alpha\beta\gamma = 0,\\
&+\alpha x + \beta y - \gamma z - \alpha\beta\gamma = 0.
\end{aligned}$$

It is remarkable that the three equations of the axis for these screws here coalesce to a single one. The screw of indeterminate pitch is thus limited,

not to a line, but to a plane. The same may be said of each of the other three screws of indeterminate pitch; they also are each limited to a plane found by giving variety of signs to the radicals in the equations just written. We have thus discovered that the complete locus of the screws of a three-system consists, not only of the family of quadrics, which contain the screws of real or imaginary, but definite pitch, but that it also contains a tetrahedron of four imaginary planes, each plane being the locus of one of the four screws of indefinite pitch.

§ 8. **Relation of the Four Planes to the Quadrics.**—The planes have an interesting geometrical connexion with the family of quadrics, which we shall now develop. The first theorem to be proved is, that each of the quadrics touches each of the planes. This is geometrically obvious, inasmuch as each quadric contains all the screws of the system which have a given pitch p; but each of the planes contains a system of screws of every pitch, among which there must be one of pitch p. There will thus be a ray in the plane, which is also a generator of the hyperboloid—but this, of course, requires that the plane be a tangent to the hyperboloid.

It is easy to verify this by direct calculation.

Write the quadric,

$$(p_1-p)\,x^2+(p_2-p)\,y^2+(p_3-p)\,z^2+(p_1-p)(p_2-p)(p_3-p)=0.$$

The tangent plane to this, at the point x', y', z', is

$$(p_1-p)\,xx'+(p_2-p)\,yy'+(p_3-p)\,zz'+(p_1-p)(p_2-p)(p_3-p)=0.$$

If we identify this with the equation

$$\alpha x + \beta y + \gamma z - \alpha\beta\gamma = 0$$

we shall obtain

$$x' = -\frac{(p_2-p)(p_3-p)}{(p_3-p_1)^{\frac{1}{2}}(p_1-p_2)^{\frac{1}{2}}}$$

$$y' = -\frac{(p_3-p)(p-p)}{(p_1-p_2)^{\frac{1}{2}}(p_2-p_3)^{\frac{1}{2}}},$$

$$z' = -\frac{(p_1-p)(p_2-p)}{(p_2-p_2)^{\frac{1}{2}}(p_3-p_1)^{\frac{1}{2}}},$$

and as these values satisfy the equation of the quadric, the theorem has been proved.

We thus see that the family of quadric surfaces are all inscribed in a common tetrahedron, but the family of quadrics have four common points, as well as four common tangent planes. For, write the two cones

$$x^2 + y^2 + z^2 = 0,$$

$$p_1 x^2 + p_2 y^2 + p_3 z^2 = 0.$$

These cones have four generators in common, and the four points in which these generators cut the plane at infinity will lie on every surface of the type

$$(p_1 - p)x^2 + (p_2 - p)y^2 + (p_3 - p)z^2 + (p_1 - p)(p_2 - p)(p_3 - p) = 0.$$

We now see the distribution of the screws in the imaginary planes. In each one of these planes there are a system of parallel lines; each line of this system passes through the same point at infinity, which is, of course, one of the four points just referred to. Every line of the parallel set, when it receives appropriate pitch, belongs to the three-system.

It thus appears that the ambiguity in the pitches of the screws in the planes is only apparent. The system of screw co-ordinates which usually defines a screw with absolute definiteness, loses that definiteness for the screws in these planes. Each plane contains a whole pencil of screws, radiating from a point at infinity, but our co-ordinates can only represent these screws collectively, for the three co-ordinates then represent, not a single screw, but a whole pencil of screws. As the pitches vary on every screw of the pencil, the co-ordinates can only meet this difficulty by representing the pitch as indeterminate.

The proof that only a single screw of each pitch is found in the pencil is easily given. If there were two, then the same hyperboloid would have two generators in this plane of equal pitch; but this is impossible, because, from the known properties of the three-system, only one of these generators belongs to the three-system, and the other to the reciprocal system.

§ 9. **The Pitch Conics.**—The discussion in the last article will prepare

us for the plane representation of the screws of given pitch p, for we have

$$p(\theta_1^2 + \theta_2^2 + \theta_3^2) - p_1\theta_1^2 - p_2\theta_2^2 - p_3\theta_3^2 = 0.$$

This, of course, represents a conic section, and, accordingly, we have the following theorem :—

The locus of points corresponding to screws of given pitch is a conic section.

A special case is that where the pitch is zero, in which case the locus is given by

$$p_1\theta^2 + p_2\theta_2^2 + p_3\theta_3^2 = 0.$$

This we shall often refer to as the *zero-pitch conic.*

Another important case is that where p is infinite, in which case the equation is

$$\theta_1^2 + \theta_2^2 + \theta_3^2 = 0.$$

The zero-pitch conic and the infinite pitch conic intersect in four points, and through these four points all the other conics must pass. The points, of course, correspond to the screws of indeterminate pitch : we may call them P_1, P_2, P_3, P_4.

Any conic through these four critical points will be a conic of equal pitch screws.

As a straight line cuts a conic in two points, we see the well-known theorem, that every cylindroid will have on it two screws of given pitch.

The two principal screws on a cylindroid are those of maximum and minimum pitch ; they will be found by drawing through P_1, P_2, P_3, P_4, the two conics touching the straight line corresponding to the cylindroid. The two points of contact are the screws required.

If A and B are the two principal screws on a cylindroid, then any pair of harmonic conjugates to A and B represent a pair of screws of equal pitch.

For if $S + kS' = 0$ be a system of conics, then it is well known that the pairs of points in which a fixed ray is cut by this system form a system in involution. The double points of this involution are the points of contact of the two conics of the system which touch the line.

§ 10. **The Angle between Two Screws.**—From the equations of the screw given in § 7, we see that the direction cosines are proportional to θ_1, θ_2, θ_3; for if we take the point infinitely distant we find the equations reduce to

$$\frac{x}{\theta_1} = \frac{y}{\theta_2} = \frac{z}{\theta_3}.$$

Accordingly, the line drawn parallel to the screw through the origin has its direction cosines proportional to θ_1, θ_2, θ_3, and hence the direction cosines of the screw are

$$\frac{\theta_1}{\sqrt{\theta_1^2 + \theta_2^2 + \theta_3^2}}, \quad \frac{\theta_2}{\sqrt{\theta_1^2 + \theta_2^2 + \theta_3^2}}, \quad \frac{\theta_3}{\sqrt{\theta_1^2 + \theta_2^2 + \theta_3^2}}.$$

The cosine of the angle between two screws, θ and ϕ, will therefore be

$$\frac{\theta_1\phi_1 + \theta_2\phi_2 + \theta_3\phi_3}{\sqrt{\theta_1^2 + \theta_2^2 + \theta_3^2}\,\sqrt{\phi_1^2 + \phi_2^2 + \phi_3^2}}.$$

By the aid of the infinite pitch conic we can give to this a geometrical interpretation.

The co-ordinates of a screw on the straight line joining θ and ϕ will be

$$\theta_1 + \lambda\phi_1, \quad \theta_2 + \lambda\phi_2, \quad \theta_3 + \lambda\phi_3.$$

If we substitute this in the equation to the infinite pitch conic we obtain

$$\theta_1^2 + \theta_2^2 + \theta_3^2 + 2\lambda(\theta_1\phi_1 + \theta_2\phi_2 + \theta_3\phi_3) + \lambda^2(\phi_1^2 + \phi_2^2 + \phi_3^2) = 0.$$

Writing this in the form

$$a\lambda^2 + 2b\lambda + c = 0,$$

of which λ_1 and λ_2 are the roots, we have, as the four values of λ, corresponding, respectively, to the points θ and ϕ, and to the points in which their chord cuts the infinite pitch conic,

$$\lambda_1, \quad \lambda_2, \quad 0, \quad \infty.$$

The anharmonic ratio is

$$\frac{\lambda_1}{\lambda_2},$$

or

$$\frac{b - \sqrt{b^2 - ac}}{b + \sqrt{b^2 - ac}}.$$

If ω be the angle between the two screws, θ and ϕ, then

$$\cos\omega = \frac{b}{\sqrt{ac}},$$

and the anharmonic ratio reduces to

$$e^{-2i\omega},$$

whence we deduce the following theorem:—

The angle between two screws is equal to $\frac{1}{2}i$ times the logarithm of the anharmonic ratio in which their corresponding chord is divided by the infinite pitch conic.

The reader will be here reminded of the geometry of non-Euclidian space, in which a magnitude called the intervene analogous to the distance between two points is equal to $\frac{1}{2}i$ times the logarithm of the anharmonic ratio in which their chord is divided by the absolute. We have only to call the infinite pitch conic the absolute, and the angle between two screws is the intervene between their corresponding points.

§ 11. **Screws at Right Angles.**—If two screws, θ and ϕ, be at right angles, then

$$\theta_1\phi_1 + \theta_2\phi_2 + \theta_3\phi_3 = 0.$$

In other words, θ and ϕ are conjugate points of the infinite pitch conic,

$$\theta_1^2 + \theta_2^2 + \theta_3^2 = 0.$$

All the screws at right angles to a given screw lie on the polar of the point with regard to the infinite pitch conic. Hence we see that all the screws perpendicular to a given screw lie on a cylindroid. This is otherwise obvious, for a screw can always be found with an axis parallel to a given direction. If, therefore, a cylindroid of the system be taken, a screw of the system parallel to the nodal axis of that cylindroid can also be found, and thus we have the cylindroid and the screw, which stand in the relation of the pole and the polar to the infinite pitch conic.

A point on the infinite pitch conic must represent a screw at right angles to itself. Every straight line cuts the infinite pitch conic in two points, and thus every cylindroid has two screws of infinite pitch, and each of these

screws is at right angles to itself. This we can also prove from the equation of the cylindroid,

$$z(x^2 + z^2) - 2m\,xy = 0.$$

The two screws of infinite pitch are given by the intersection of the plane at infinity, $m = 0$, with the two planes $x \pm iy = 0$.

In general, the direction cosines of the nodal axis of a cylindroid are proportional to the co-ordinates of the pole of the line corresponding to the cylindroid with respect to the infinite pitch conic.

§ 12. **Reciprocal Screws.**—If θ_1, θ_2, θ_3 be the co-ordinates of a screw, and ϕ_1, ϕ_2, ϕ_3 those of another screw, then it is known ("Theory," chap. IV.), that the condition for these two screws to be reciprocal is

$$p_1\theta_1\phi_1 + p_2\theta_2\phi_2 + p_3\theta_3\phi_2 = 0.$$

We are thus led to the following theorem, which is of fundamental importance in the present investigation:—

A pair of reciprocal screws are conjugate points with respect to the zero-pitch conic.

From this theorem we can at once draw the following conclusions:—

All the screws of the system reciprocal to a given screw θ lie upon a cylindroid.

For the locus of points conjugate to θ is, of course, the polar of θ with respect to the zero-pitch conic, and this polar will, of course, correspond to a cylindroid.

On any cylindroid one screw can always be found reciprocal to a given screw θ. For this will, of course, be the intersection of the polar of θ with the line corresponding to the given cylindroid.

A triad of co-reciprocal screws will correspond to a self-conjugate triangle of the zero-pitch conic.

§ 13. **The Principal Screws of the System.**—Draw the zero-pitch conic A, and the infinite pitch conic B, which intersect in the four screws

of indeterminate pitch, P_1, P_2, P_3, P_4 (see fig. 2). Draw the diagonals of the complete quadrilateral, and let these intersect in the points X, Y, Z.

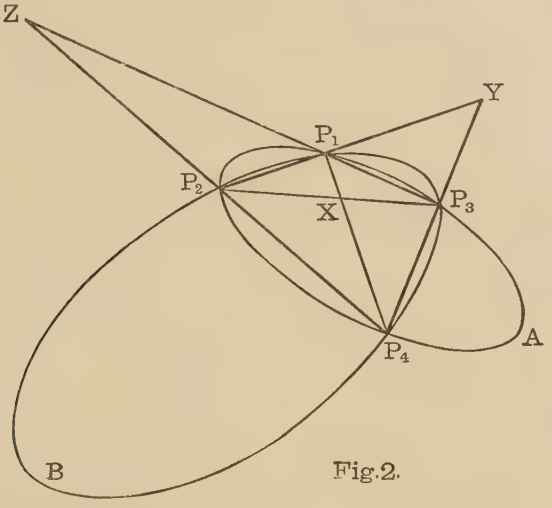


Fig. 2.

These three points are of unique interest. Take any pair of them, X and Y; then, by the known properties of conics, X and Y are conjugate points with respect to both of the conics A and B. The screws X and Y must therefore be both rectangular and reciprocal. If p_θ and p_ϕ be the pitches of these screws; if ω be their angle, and d their perpendicular distance, then the virtual coefficient is

$$(p_\theta + p_\phi) \cos \omega - d \sin \omega,$$

as they are reciprocal, this is zero, and as ω is a right angle, we must have $d = 0$; in other words, the screws corresponding to X and Y must intersect at right angles. The same may be proved of either of the pairs X and Z or Y and Z. The points X, Y, Z must therefore correspond to three screws of the system mutually perpendicular, and intersecting at a point. But in the whole system there is only a single triad of screws possessing these properties. They are the axes in the equations of § 7, and are known as the principal screws of the system. The three points X, Y, Z being the vertices of a self-conjugate triangle with respect to both the conics A and B, we have the following theorem :—

The vertices of the common conjugate triangle to the system of pitch conics correspond to the three principal screws of the three-system.

§ 14. **Expression for the Pitch.**—Each conic drawn through the four points of indeterminate pitch, P_1, P_2, P_3, P_4 is the locus of screws with a given pitch belonging to the system. We are thus led to connect the constancy of the pitch at each point of this conic with another feature of constancy, viz. that of the anharmonic ratio subtended by a variable point of the conic with the four fixed points. The connexion between the pitch and the anharmonic ratio will now be demonstrated.

Let θ_1, θ_2, θ_3 be the co-ordinates of any point on the conic, and let α, β, γ be the co-ordinates of one of the four points, say P; then if ϕ_1, ϕ_2, ϕ_3 be the current co-ordinates, the equation of the line joining θ to P_1 is

$$\begin{vmatrix} \phi_1, & \phi_2, & \phi_3 \\ \theta_1, & \theta_2, & \theta_3 \\ \alpha, & \beta, & \gamma \end{vmatrix} = 0.$$

As we are dealing with the anharmonic ratio of a pencil, we may take any section for the calculation of the ratios, and, accordingly, make

$$\phi_3 = 0\,;$$

and we have for the co-ordinates of the point in which the line joining θ and P_1 cuts $\phi_3 = 0$, the conditions,

$$\frac{\phi_1}{\phi_2} = -\frac{\theta_3\alpha - \theta_1\gamma}{\theta_2\gamma - \theta_3\beta}.$$

By changing the sign of α, and then changing the signs of β and of γ, we shall obtain the four points in which the pencil $\theta(P_1, P_2, P_3, P_4)$ cuts the axis $\phi_3 = 0$. If four values of $\frac{\phi_1}{\phi_2}$ be represented by k, l, m, n, the required anharmonic ratio is, of course,

$$\frac{(n-l)(m-k)}{(n-m)(l-k)},$$

and after a few reductions we find this reduces to

$$\theta(P_1P_2P_3P_4) = \frac{\beta^2(\gamma^2\theta_1^2 - \alpha^2\theta_3^2)}{\alpha^2(\gamma^2\theta_2^2 - \beta^2\theta_3^2)}.$$

But we have

$$p = \frac{p_1^2\theta^2_1 + p_2\theta_2^2 + p_3\theta_3^2}{\theta_1^2 + \theta_2^2 + \theta_3^2}.$$

Eliminating θ_1^2, we find that θ_2^2 and θ_3^2 disappear also when we make

$$\alpha^2 = p_2 - p_3, \quad \beta^2 = p_3 - p_1, \quad \text{and} \quad \gamma^2 = p_1 - p_2,$$

and we obtain the following result :

$$\theta(P_1P_2P_3P_4) = \frac{p_3 - p_1}{p_3 - p_2} \times \frac{p_2 - p}{p_1 - p},$$

which gives the following theorem :—

Measure off distances p_1, p_2, p_3, p, from an arbitrary point on a straight line, then the anharmonic ratio of the four points thus obtained is equal to the anharmonic ratio subtended by any point of the p-pitch conic at the four points of indeterminate pitch.

IV. The Special System in which one of the Conics is a Circle.

§ 15. **The Zero-pitch Conic a Circle.**—It will be possible and consistent with perfect generality to take a circle as the conic zero-pitch. For let us take three angles, A, B, C, such that their sum is equal to π, and such that the equations,

$$\frac{\sin 2A}{p_1} = \frac{\sin 2B}{p_2} = \frac{\sin 2C}{p_3}$$

are satisfied when p_1, p_2, p_3 are the principal pitches. We then draw any triangle with angles A, B, C, and the equation

$$\alpha_1^2 \sin 2A + \alpha_2^2 \sin 2B + \alpha_3^2 \sin 2C = 0$$

is the equation of the conic of zero pitch. But this conic is known to be a circle with its centre at the intersection of perpendiculars from the vertices of the triangle on the opposite sides.

We thus have, as the system of pitch conics,

$$\alpha_1^2 \sin 2A + \alpha_2^2 \sin 2B + \alpha_3^2 \sin 2C - p(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) = 0.$$

The centre of a conic is the pole of the line at infinity. The polar of α_1', α_2', α_3' is

$$\alpha_1\alpha_1'(\sin 2A - p) + \alpha_2\alpha_2'(\sin 2B - p) + \alpha_3\alpha_3'(\sin 2C - p) = 0,$$

and the line at infinity is

$$\alpha_1 \sin A + \alpha_2 \sin B + \alpha_3 \sin C = 0,$$

whence, for the centre of the conics we have the co-ordinates

$$\frac{\sin A}{\sin 2A - p}, \quad \frac{\sin B}{\sin 2B - p}, \quad \frac{\sin C}{\sin 2C - p}.$$

From this we easily find that the equation to the locus of the centres of the systems of conics is

$$0 = \sin 2A \sin(B - C)\,\theta_2\theta_3 + \sin 2B \sin(C - A)\,\theta_3\theta_1 + \sin 2C \sin(A - \theta)\,\theta_1\theta_2.$$

This conic circumscribes the triangle of reference. It must also pass through the centre of the pitch circle; but we have already seen that the centre of this circle lies at the intersections of the perpendiculars from the vertices of the triangle of reference on the opposite sides. Hence this conic must pass through this intersection also. But it is known that a conic which passes through the vertices of a triangle, and also through the intersection of its axes, is an equilateral hyperbola, and hence we have the theorem thus stated:—

The locus of the centres of the system of pitch conics is an equilateral hyperbola.

The connexion between the pitch of any conic and the dimensions of its axes will now be found. We shall find this by calculating the anharmonic ratio in which the conic divides the line joining the two circular points at infinity. Let a and b be the semiaxes of this ellipse, then the equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0,$$

$$x^2 + y^2 = 0,$$

will define the two pairs constituting the pencil of four rays of which the anharmonic ratio is required. That ratio is easily seen to be

$$\frac{(a-b)^2}{(a+b)^2}.$$

We can construct this ratio in a different way.

The equation of the pair of rays from C to the pair of circular points is

$$\alpha_1^2 + 2\alpha_1\alpha_2 \cos C + \alpha_2^2 = 0.$$

The equation of the pair of rays from C to the intersection of the line at infinity,

$$\alpha_1 \sin A + \alpha_2 \sin B + \alpha_3 \sin C = 0,$$

with the conic,

$$\alpha_1^2 \sin 2A + \alpha_2^2 \sin 2B + \alpha_3^2 \sin 2C - p(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) = 0,$$

is found, by eliminating α_3, to be of the form

$$A\alpha_1^2 + 2B\alpha_1\alpha_2 + C\alpha_2^2 = 0:$$

with a few reductions we find, finally, from identifying the two values of the anharmonic ratio,

$$\left(\frac{ab}{a^2+b^2}\right)^2 = \frac{4m^2 - 6mp + np^2}{(4m-3p)^2};$$

or it may be written

$$\frac{a^2-b^2}{a^2+b^2} = \sqrt{9-4n}\,\frac{p}{4m-3p},$$

where

$$m = \sin A \sin B \sin C,$$

$$n = \sin^2 A + \sin^2 B + \sin^2 C,$$

$$= 2 + 2\cos A \cos B \cos C,$$

$$9 - 4n = 1 - 8\cos A \cos B \cos C.$$

If $p = 0$, we see that $a = b$, as it obviously should. If $a^2 + b^2 = 0$, then

$$p = \tfrac{4}{3} \sin A \sin B \sin C,$$

which is the pitch of the only equilateral hyperbola belonging to the system.

For the parabolas of the system we have the pitches defined by

$$4m^2 - 6mp + np^2 = 0.$$

In general there are two conics of any given eccentricity belonging to the system.

§ 16. **The Equilateral Hyperbola which is the Locus of the Centres of the Pitch Conics.**—We have already seen that the equilateral hyperbola, which is the locus of the centres, passes through the three vertices of the triangle of reference and the intersection of its three perpendiculars. We are now able to add a fifth point, also on the curve, possessing geometrical significance. The co-ordinates of a point on the hyperbola are, generally,

$$\frac{\sin A}{\sin 2A - p}, \quad \frac{\sin B}{\sin 2B - p}, \quad \frac{\sin C}{\sin 2C - p}.$$

If $p = \infty$, we have as the co-ordinates of the centre of the conic of infinite

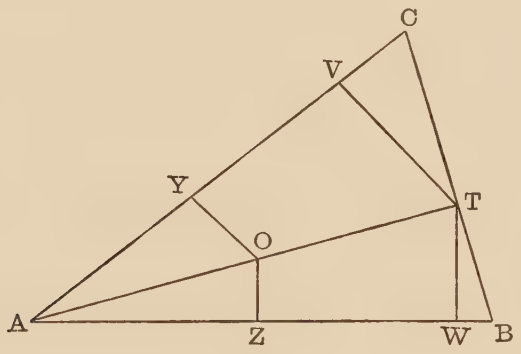


Fig.3.

pitch the symmedian point, $\sin A$, $\sin B$, $\sin C$. This point is the centre of gravity of particles a^2, b^2, c^2 situated at the vertices A, B, C, for

$$\frac{\beta}{\lambda} = \frac{OY}{OZ} = \frac{TY}{TW} = \frac{CT \sin C}{BT \sin B};$$

but if

$$\frac{\beta}{\lambda} = \frac{\sin B}{\sin C},$$

we have

$$\frac{CT}{BT} \frac{\sin^2 B}{\sin^2 C} = \frac{b^2}{c^2},$$

whence the result just stated.

§ 17. **On the Relation between the Pitch of the Screws on a Conic of the System, and the Position of the Centre of that Conic on the Equilateral Hyperbola.**—Considering that there is one conic for each pitch, it might be expected that the relation in question was of the one-to-one character. The proper method for making the transition from the trilinear to the rectangular co-ordinates is to utilize the anharmonic property of four points on a conic.

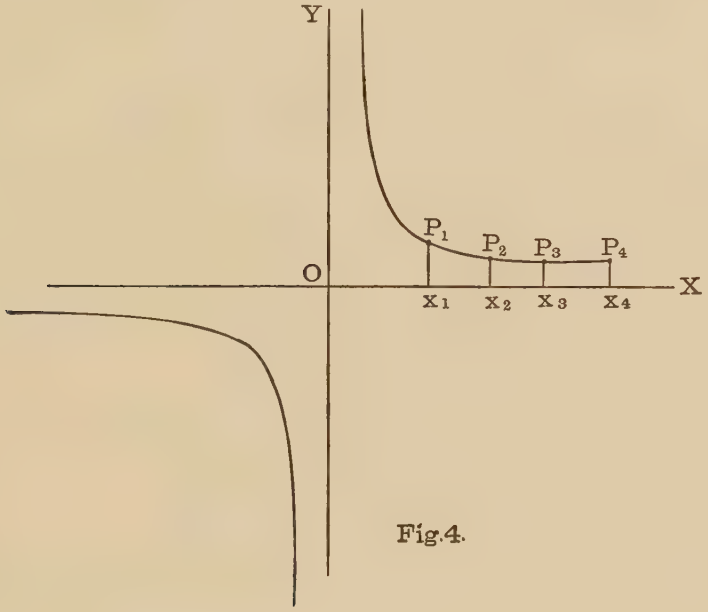


Fig. 4.

Let OX and OY, in the adjacent figure, be the two asymptotes of the rectangular hyperbola, of which the equation is

$$xy = h^2.$$

Let P_1, P_2, P_3, P_4 be four points on this conic; they will, of course, subtend a constant anharmonic ratio at *any* point of the curve. I take this to be the point of contact at infinity made by the asymptote OY. The pencil will thus consist of the four ordinates through P_1, P_2, P_3 P_4, and the anharmonic ratio of the group is therefore

$$\frac{(x_4 - x_2)(x_3 - x_1)}{(x_4 - x_3)(x_2 - x_1)}.$$

The four points, P_1, P_2, P_3, P_4 are quite arbitrary, and we shall choose three of them to be the vertices of the triangle of reference through which it is known that the hyperbola passes. The fourth point shall be that which is the locus of the screws of pitch p. Thus the trilinear co-ordinates of the four points P_1, P_2, P_3, P_4 are as follows:—

P_1	1	0	0
P_2	0	1	0
P_3	0	0	1
P_4	$\frac{\sin^2 A}{\sin 2A - p}$,	$\frac{\sin B}{\sin 2B - p}$,	$\frac{\sin C}{\sin 2C - p}$.

For the calculation of the anharmonic ratio of these four points we may choose *any* point known to be on the curve for the vertex of the pencil. I shall take the centre of the conic of infinite pitch screws as this point, which we have already seen to have for co-ordinates

$$\sin A, \quad \sin B, \quad \sin C.$$

By a brief calculation the anharmonic ratio is found to be

$$\frac{(\sin 2A - \sin 2C)(\sin 2B - p)}{(\sin 2B - \sin 2C)(\sin 2A - p)}.$$

Identifying this with the former value, and taking x_1 as the current co-ordinate corresponding to p, we obtain

$$p = \frac{a + bx}{c + dx},$$

where a, b, c, d are all constants. We have thus proved a homographic relation between each P and the corresponding x.

A certain pair of relations between a, b, c, d can be easily obtained. For the pitches of the two parabolas belonging to the system must be $a \div c$ and $b \div d$, respectively, because the centres of the conics are then at infinity, and therefore must be given by $x = 0$ and $x = \infty$, respectively. We have, however, already seen that the pitches on the parabola are given by the equations

$$np^2 - 6mp + 4m^2 = 0,$$

whence we have

$$na^2 - 6mac + 4m^2c^2 = 0,$$

$$nb^2 - 6mbd + 4m^2d^2 = 0.$$

§ 18. **Co-ordinates of the Centre of the Equilateral Hyperbola.**—The centre is the pole of the line at infinity with respect to the curve. Now, the conic may be written in the form

$$\sin 2A \sin(B-C)\,\beta\gamma + \sin 2B \sin(C-A)\,\alpha\gamma + \sin 2C \sin(A-B)\,\alpha\beta = 0;$$

so that, if α, β, γ be the co-ordinates of the pole of

$$\alpha \sin A + \beta \sin B + \gamma \sin C = 0,$$

we have

$$H \sin A = \sin 2B \sin(C-A)\,\gamma + \sin 2C \sin(A-B)\,\beta,$$

$$H \sin B = \sin 2C \sin(A-B)\,\alpha + \sin 2A \sin(B-C)\,\gamma,$$

$$H \sin C = \sin 2A \sin(B-C)\,\beta + \sin 2B \sin(C-A)\,\alpha;$$

whence

$$\alpha = \cos A \sin^2(B-C),$$

$$\beta = \cos B \sin^2(C-A),$$

$$\gamma = \cos C \sin^2(A-B).$$

The centre of the inscribed circle and the three centres of the exscribed triangles must all belong to one hyperbola of the system, for

$$p = \frac{\alpha_1^2 \sin 2A + \alpha_2^2 \sin 2B + \alpha_3^2 \sin 2C}{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}$$

will give the same value for p if we have, for α_1, α_2, α_3, either of the values

$$\pm 1, \quad \pm 1, \quad \pm 1.$$

In this case

$$3p = \sin 2A + \sin 2B + \sin 2C,$$

or

$$= \tfrac{4}{3} \sin A \sin B \sin C.$$

We have already seen that this is an equilateral hyperbola, and the only one belonging to the system.

V. DETAILS OF THE NUMERICAL ILLUSTRATION PRESENTED IN THE ACCOMPANYING PLATE.

§ 19. **Transformation to Rectangular Co-ordinates.**—It will be convenient to transform from trilinear to ordinary rectangular co-ordinates even at some sacrifice of symmetry.

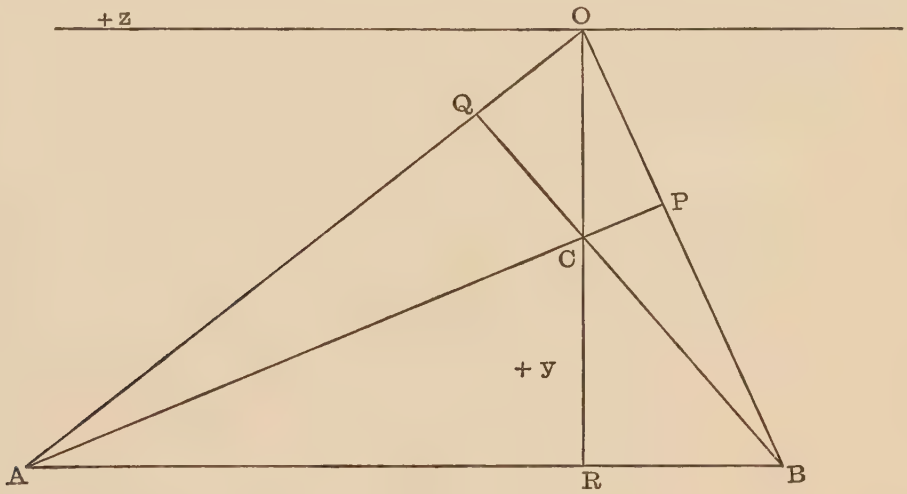


Fig.5.

Let ABC (fig. 5) be the triangle, and O the orthocentre, and let D be the diameter of the circumscribing circle. A line through O, parallel to AB, is the axis of x, and OR is the axis of y, then α, β, γ being the perpendiculars on the sides, we have

$$x = \alpha \operatorname{cosec} B + \gamma \cot B - D \sin A \cos B,$$

$$y = D \cos A \cos B - \gamma,$$

and also

$$\alpha \sin A + \beta \sin B + \gamma \sin C = D \sin A \sin B \sin C.$$

Making these substitutions, the equation to the system of conics is

$$+ x^2 \left(\sin^2 A + \sin^2 B - \frac{2 \sin A \sin B \sin C}{p} \right),$$

$$+ y^2 \left(\cos^2 A + \cos^2 B + 1 - \frac{2 \sin A \sin B \sin C}{p} \right),$$

$$+ xy\,(\sin 2B - \sin 2A),$$

$$+ Dx(\sin 2B \cos C - \sin 2A \cos C),$$

$$+ Dy\,(2\cos^2 B \cos C + 2\cos^2 A \cos C - 2\cos A \cos B),$$

$$+ D^2\left(\cos^2 B \cos^2 C + \cos^2 A \cos^2 C + \cos^2 A \cos^2 B - \frac{2 \sin A \sin B \sin C \cos A \cos B \cos C}{p}\right).$$

§ 20. **Substitution of Actual Numbers.**—In drawing the figure shown on the accompanying Plate, I have taken the following values for the sides:

$$a = 117, \quad b = 189, \quad c = 244.$$

These dimensions are in millimeters, and the system of conics was drawn on papier millimetrique. The equation to the system is—

$$0 = + x^2\left(\cdot 7864115 - \frac{\cdot 6852041}{p}\right) + y^2\left(2\cdot 2135882 - \frac{\cdot 6852041}{p}\right)$$

$$+ \cdot 1649571xy - 9\cdot 46700x - 430\cdot 5179y + 25199\cdot 54 + \frac{5726\cdot 031}{p} = 0.$$

Among the critical values we have the following:—

1. The two parabolas of the system for which the pitches are, respectively, ·8766003 and ·3088817.
2. The three cases in which the conic breaks into a pair of straight lines for the pitches $\sin 2A$, $\sin 2B$, $\sin 2C$, respectively. Of these, the first alone is a real pair corresponding to the pitch ·8256034. The equations of these lines are

$$x = \quad 7\cdot 84y - 978,$$

$$x = - 4\cdot 06y + 760.$$

For convenience in plotting, the current co-ordinates on each conic are expressed by means of an auxiliary angle; thus, for example, in computing points on the hyperbola with the pitch ·748984, I used the equations

$$x = 66 + 26\cdot 1 \sec\theta + 132 \tan\theta,$$

$$y = 161\cdot 5 + 40\cdot 7 \sec\theta,$$

while the ellipse with pitch ·9 may be readily constructed from the equations

$$x = -367 - 168\cos\theta \pm 351\sin\theta,$$

$$y = +169 + 51\cos\theta.$$

The following are the pitches of the several curves drawn in the Plate :—

Vertices of the triangle, + ·992, + ·825, − ·448,

Ellipses, + ·96, + ·92, + ·1, − ·4,

Parabola, + ·309,

Hyperbola, + ·748984, + ·8300467.

The equilateral hyperbola passing through the orthocentre, the symmedian point and the vertices of the triangle of reference is the locus of the centres of all the conics of the system.

VI. Screws Represented by Points on a System of Coaxial Circles.

§ 21. **Projection of the System of Cones into Coaxial Circles.**—The three-system can be represented by a system of conics with four points in common. In another plane we can draw a system of coaxial circles (with imaginary radical axis), and we can establish a point-by-point correspondence between the coaxial circles and the system of conics. The two systems are projective, and we obtain the result thus stated.

The screws of a three-system may be represented by the points on a series of coaxial circles. The following are the chief points in this correspondence :—

1. All the screws of a given pitch in the three-system correspond to the points on the circumference of one of the coaxial circles.
2. Let a, b, c be the three principal pitches, then the radical axis of the two circles and the line at infinity constitute the locus of points corresponding to pitch b.
3. The foci of the confocal system correspond to the screws of pitches a and c.

We first note the general principle by which a correspondence in one plane, where Cartesian co-ordinates are used, is to be calculated for another plane where trilinear co-ordinates are used. Let α, β, γ be the trilinear co-ordinates of a point P. Then every function or property of this point must admit of complete specification by the aid of α, β, γ. Among these functions of P is the x of its correspondent in the second plane. We must have

$$x = f(\alpha, \beta, \gamma);$$

but all functions of trilinear co-ordinates must be homogeneous; and as the correspondence is to be of the one-to-one type, we must have a linear equation in α, β, γ corresponding to $x = 0$. We therefore find that

$$x = \frac{l\alpha + m\beta + n\gamma}{\Omega},$$

and, similarly,

$$y = \frac{l'\alpha + m'\beta + n'\gamma}{\Omega'}.$$

Ω and Ω' are linear functions of α, β, γ, and as an equation of the first degree in x and y must give a linear equation, we must have

$$\Omega = \Omega';$$

otherwise a straight line in the second plane would have as its correspondent a conic in the first. The equation $\Omega = 0$ denotes that it is to be the line in the first plane corresponding to the line at infinity in the second.

We shall take the radical axis and the line of centres as the axes of y and x, respectively.

The system of conics is

$$a^2 + b^2 + c^2 - p(\alpha^2 + \beta^2 + \gamma^2) = 0.$$

These form a pair of straight lines which are real when, and only when

$$p = b,$$

$$(a - b)\alpha^2 - (b - c)\gamma^2 = 0;$$

or, making

$$\frac{b-c}{a-b}=h^2.$$

The only pair of real lines are

$$\alpha \pm h\gamma = 0.$$

These must correspond to a pair of real lines in the second plane; but the only case in which one of the coaxial circles becomes decomposed into a pair of straight lines is when one of the lines is the radical axis, and the other is the line at infinity. Hence we must have the axis of y corresponding to

$$\alpha - h\gamma,$$

and the line at infinity corresponding to

$$\alpha + h\gamma,$$

and hence

$$x = m\,\frac{\alpha - h\gamma}{\alpha + h\gamma}.$$

There are two other cases in which the pitch conic splits into a pair of straight lines, viz. where $p = a$ and $p = c$, respectively. In these cases both straight lines are imaginary. They each have a real vertex, $\beta = 0$, $\gamma = 0$ in one case, and $\alpha = 0$, $\beta = 0$ in the other.

In the second plane we find the foci of the system of coaxial circles are each the vertices of the corresponding pair of imaginary straight lines. These each lie on the axis of x. Therefore, whatever y may be, it must vanish if $\alpha = 0$ and $\beta = 0$, or if $\beta = 0$ and $\gamma = 0$. Hence we obtain

$$\left.\begin{aligned} x &= m\,\frac{\alpha - h\gamma}{\alpha + h\gamma}, \\ y &= n\,\frac{\beta}{\alpha + h\gamma}, \end{aligned}\right\} \text{ where } h^2 = \frac{b-c}{a-b},$$

where m and n are constants which must be submitted to another relation. This is to be found by expressing that, when for α, β, γ we substitute their values in x and y in the equation

$$a\alpha^2 + b\beta^2 + c\gamma^2 - p(\alpha^2 + \beta^2 + \gamma^2) = 0,$$

we shall obtain a system of circles: we have

$$\alpha = \gamma h \frac{m+x}{m-x},$$

$$\beta = \gamma \frac{2hmy}{n(m-x)}.$$

Substituting, we find

$$n^2a\gamma^2h^2(m+x)^2 + 4b\gamma^2h^2m^2y^2 + cn^2(m-x)^2\gamma^2$$

$$-p(n^2\gamma^2h^2(m+x)^2 + 4\gamma^2h^2m^2y^2 + n^2(m-x)^2\gamma^2).$$

Let us make $$4m^2(b-c) = n^2(a-c),$$

and then, after a few reductions, we find

$$\left(x + m\frac{(a+c)(b+p)-2(ac+bp)}{(a-c)(b-p)}\right)^2 + y^2 = R^2,$$

where

$$R^2 = \frac{4m^2(a-b)(b-c)}{(a-c)^2}.\frac{(a-p)(p-c)}{(b-p)^2}.$$

This fulfils the obvious tests of vanishing if p be either a or c, and becoming infinite if $p = b$.

The centre of the circle is at a distance from the origin

$$m\frac{(c-b)(p-a)+(a-b)(p-c)}{(a-c)(b-p)},$$

and m is the distance from the origin to the two foci on either side. An interesting verification of this theory can now be given.

The intercept on the radical axis by all the circles is

$$2im.$$

The sign of the angle ϕ subtended in the circle of radius R by this arc is

$$i\frac{m}{R},$$

whence we have

$$\sin^2\phi = -\frac{(a-c)^2(b-p)^2}{4(a-b)(b-c)(a-p)(p-c)},$$

and by a few reductions

$$\cos^2\phi = \frac{((c-b)(p-a)+(a-b(p-c))^2}{4(a-b)(b-c)(a-p)(p-c)}.$$

The anharmonic ratio of the pencil subtended by two rays intersecting on the circumference at the angle ϕ, and the rays from this circumference to the circular point is

$$\frac{\cos\phi + i\sin\phi}{\cos\phi - i\sin\phi}.$$

Introducing these values for $\sin\phi$ and $\cos\phi$, we find for the arharmonic ratio

$$\frac{(b-c)(p-a)}{(b-a)(p-c)}.$$

Thus verifying the theorem previously obtained in the general case, which states that the function last written is the anharmonic ratio of the pencil subtended by the four common points of

$$a\alpha^2 + b\beta^2 + c\gamma^2 - p(\alpha^2+\beta^2+\gamma^2) = 0$$

at any point of the cone.

We have now a construction for finding the centre of a circle with a given pitch belonging to the system.

From the formula for the distance x of the centre of the circle belonging to pitch p from the centre, we have

$$x = m\frac{(c-b)(p-a)+(a-b)(p-c)}{(a-c)(b-p)}.$$

This shows that all the x points are homographic with the several corresponding values of p.

x,	p,
$+m$,	c,
$-m$,	a,
∞,	b.

If, therefore, we can place two ranges with these three pairs in perspective, we shall have solved the problem

$$HA = c, \quad HC = a, \quad HK = b,$$

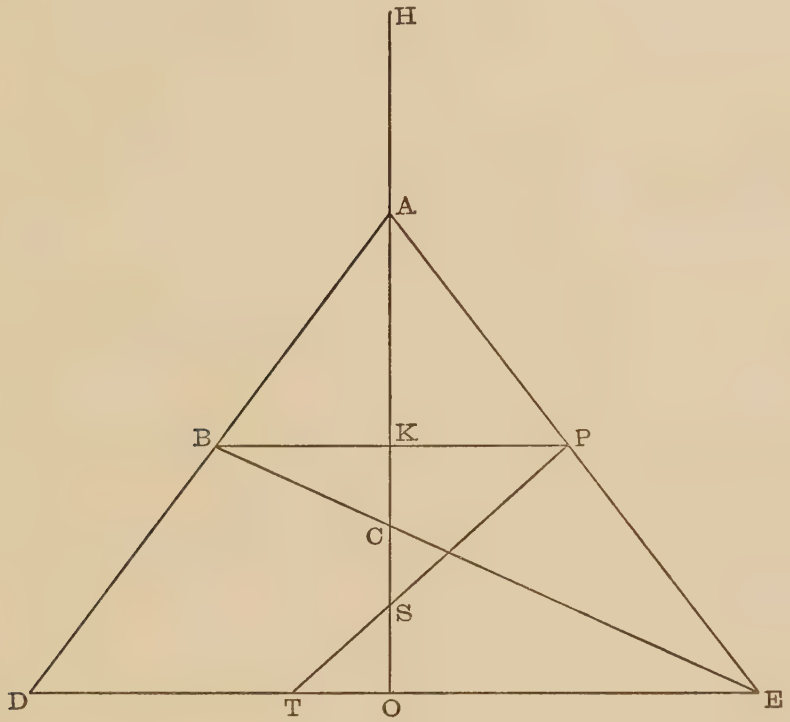


Fig. 6.

AKCO is an harmonic section which gives the position of *H*: any transversal, such as *PST* gives *T* as the centre of the circle with pitch *HS*.

VII. Applications to the Theory of Impulsive Forces.

§ 22. **On Plane Homographic Systems.**—It will be convenient first to state the leading principles of the purely geometrical theory of homography. It will then appear how exactly this theory is adapted to explain the dynamical phenomena of impulsive forces in the three-system.

Let α be any point in a plane, and let β be a corresponding point. Let us further suppose that the correspondence is of the one-to-one type, so that when one α is given then one β is known, when one β is given then can it only be the correspondent of a single α. The relation is not generally an interchangeable one. It will only be true in very special circumstances: that β, regarded as in the first system, will correspond to α in the second system.

The general relation between the points α and β will be expressed by the following equations, where α_1, α_2, α_3 are the co-ordinates of α, and β_1, β_2, β_3, the co-ordinates of β.

$$\beta_1 = (11)\,\alpha_1 + (12)\,\alpha_2 + (13)\,\alpha_3,$$

$$\beta_2 = (21)\,\alpha_1 + (22)\,\alpha_2 + (23)\,\alpha_3,$$

$$\beta_3 = (31)\,\alpha_1 + (32)\,\alpha_2 + (33)\,\alpha_3.$$

In these expressions (11), (12), &c., are the constants defining the particular character of the homographic system.

There are in general three points, which coincide with their correspondents, obtained by making

$$\beta_1 = \rho\alpha_1\,; \quad \beta_2 = \rho\alpha_2\,; \quad \beta_3 = \rho\alpha_3.$$

Introducing these values, and eliminating α_1, α_2, α_3, we obtain the following equation for ρ:—

$$0 = \begin{vmatrix} (11) - \rho, & (12), & (13) \\ (21), & (22) - \rho, & (23) \\ (31), & (32), & (33) - \rho \end{vmatrix}.$$

If we choose these three points of the vertices of the triangle of reference, the equations relating y with x assume the simple form,

$$\beta_1 = f_1\alpha_1\,; \quad \beta_2 = f_2\alpha_2\,; \quad \beta_3 = f_3\alpha_3,$$

where f_1, f_2, f_3 are three new constants.

§ 23. **A Geometrical Method of Finding Pairs of Corresponding Points.**—The equations last written suggest a simple geometrical method by which, when one point is given, its correspondent can be neatly ascertained. We assume, for this purpose, a conic U arbitrary in every respect save that the triangle of reference is to be self-conjugate with respect to it. This conic has for its equation

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = 0, \qquad U$$

in which a_1, a_2, a_3 remain quite undisposed of. We might, indeed, have taken these quantities to be, respectively, equal to the sines of double the angles of the triangle of reference. In this case U would have become a circle, and the rest of our theory would still remain perfectly general. We shall not, however, here introduce any specialization whatever of these constants.

When U has been chosen, another conic V is definitely constructed. The equation of V is

$$\frac{a_1}{f_1}x_1^2 + \frac{a_2}{f_2}x_2^2 + \frac{a_3}{f_3}x_3^2 = 0. \qquad V$$

This pair of conics suffice for the determination of the entire correspondence in consequence of the following theorem :—

The polar of any point α with regard to the conic U is the polar of the corresponding point β with regard to the conic V.

The proof is obvious, inasmuch as the polar of α with regard to U is

$$a_1\alpha_1x_1 + a_2\alpha_2x_2 + a_3\alpha_3x_3,$$

but this may be written

$$\frac{a_1}{f_1}f_1\alpha_1x_1 + \frac{a_2}{f_2}f_2\alpha_2x_2 + \frac{a_3}{f_3}f_3\alpha_3x_3 = 0,$$

which is

$$\frac{a_1}{f_1}\beta_1x_1 + \frac{a_2}{f_2}\beta_2x_2 + \frac{a_3}{f_3}\beta_3x_3 = 0,$$

and, of course, this is the polar of β with regard to the conic V.

We may state the theorem of the present article in a different manner, as follows :—

The pole of any ray with respect to U, and its pole with respect to V, are pair of corresponding points.

To avoid confusion, let it be understood that each point in the plane may be considered as belonging to either the first system or to the second system. We denote the point by α when we think of it in the first system, while we denote the point by β when it is thought of in the second system.

All the α-points on a ray A have as their correspondents the β-points on a ray B. It is required to find B.

The polars of the α-points on A, with respect to U, will all pass through the point P, which is the pole of A with respect to U. The polar of P, with respect to V, must be the required ray B.

All the α-points on a given conic A have as their correspondents the β-points on a conic B. It is required to find B.

Draw the reciprocal conic of A with regard to U, and of this reciprocal conic find the reciprocal with regard to V. This will be B.

The centre of U as an α-point, and the centre of V as a β-point, are a corresponding pair.

For each centre is the pole of the line at infinity with respect to the corresponding conic. The corespondence will form an *involution* when two points α and β will correspond, not only when α is in the first system and β in the second, but also when α is in the second system and β in the first. The relation between the conics U and V is in this case of a very special type.

We must have, not only

$$\beta_1 = f_1\alpha_1\,; \quad \beta_2 = f_2\alpha_2\,; \quad \beta_3 = f_3\alpha_3,$$

but also

$$\alpha_1 = f_1\beta_1\,; \quad \alpha_2 = f_2\beta_2\,; \quad \alpha_3 = f_3\beta_3,$$

whence

$$f_1^2 = f_2^2 = f_3^2.$$

These will be satisfied if

$$f_1 = -1\,; \quad f_2 = +1\,; \quad f_3 = +1,$$

which is the only case that need be considered.

The conics U and V then become

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = 0,$$

$$-a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = 0.$$

These equations denote a pair of conics having double contact with each other, and still further specialized by the fact that any ray through the pole of the common chord is cut harmonically by the two conics. One of the conics is imaginary.

The special nature of the system is apparent when they are projected into two concentric circles, for then the sum of the squares of the radii of these circles must be zero.

It should be noticed that if in any one case a pair of correspondents are interchangeable (other than any one of the three coincident pairs forming the double points), then, in every case, the law of interchangeability will obtain.

§ 24. **On the Cross-relations of Corresponding Points.**—Let α and β be one pair of correspondents, and let α' and β' be another pair. Then we have the following theorem, in which we employ the usual language in speaking of two points as conjugate to a conic when each lies on the polar of the other :—

If α and β' be U-conjugate, then will β and α' be U-conjugate, and conversely.

DEM.—The condition that α and β' shall be U-conjugate is

$$a_1 f_1 a_1 a_1' + a_2 f_2 a_2 a_2' + a_3 f_3 a_3 a_3' = 0 :$$

but this is equally the condition that β and α' shall be U-conjugate.

The same equation also expresses the condition that β and β' shall be V-conjugate.

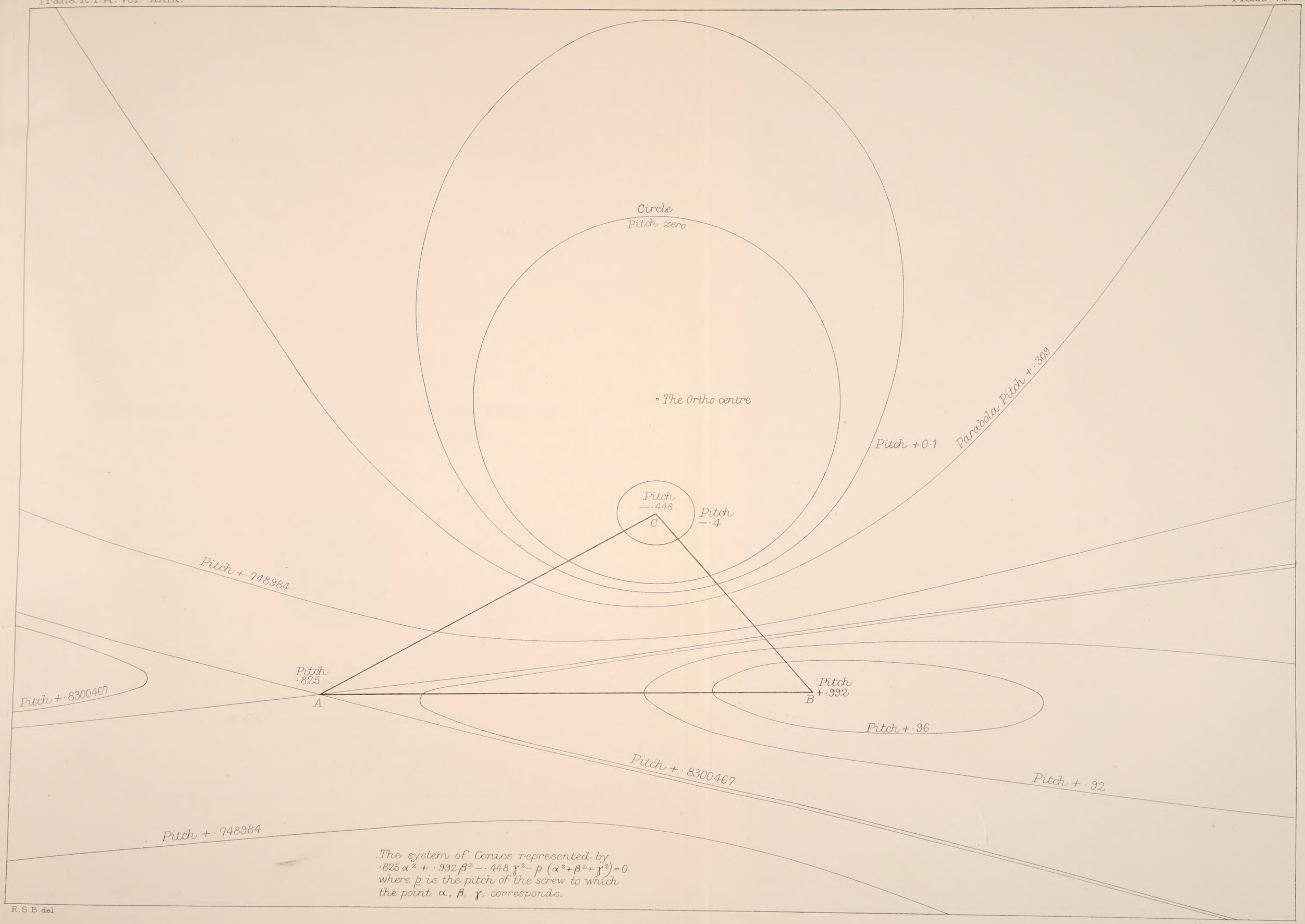
If the two α's of two corresponding pairs be U-conjugate, then the pairs may be said to be U-conjugate,or if the two β's of two corresponding pairs be V-conjugate, then the pairs may be said to be V-conjugate, and we have the following two theorems :—

If two pairs be U-conjugate, then both cross-pairs are V-conjugate.

If two pairs be V-conjugate, then both cross-pairs are U-conjugate.

These theorems have a dynamical significance.

§ 25. **Application to Impulsive and Instantaneous Screws.**—The points corresponding to impulsive screws and the corresponding instantaneous screws, form two homographic systems to which the preceding theory applies. The two ellipses are respectively the locus of the screws of zero pitch, and the imaginary ellipse representing the locus of the screws about which a body twists with zero energy. The intensity of the impulsive wrench adequate to produce the unit twist velocity on the corresponding instantaneous screw is proportional to the ratio of perpendiculars from the impulsive, and the instantaneous screw on a side of the triangle formed by the three principal screws.



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